CS103
Winter 2020

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Problem Set 2

This second problem set explores mathematical logic and dives deeper into formal mathematical proofs. We've chosen the questions here to help you get a more nuanced understanding for what first-order logic statements mean (and, importantly, what they don't mean) and to give you a chance to practice your proofwriting. By the time you've completed this problem set, we hope that you have a much better grasp of mathematical logic and how it can help improve your proofwriting structure.

Before attempting this problem set, we recommend that you do the following:

- Familiarize yourself with the online Truth Table Tool and play around with it a bit to get a feel for the propositional connectives.
- Read the online "Guide to First-Order Translations."
- Read "First-Order Translation Checklist," to get a better sense for common errors in first-order logic translations and how to avoid them.

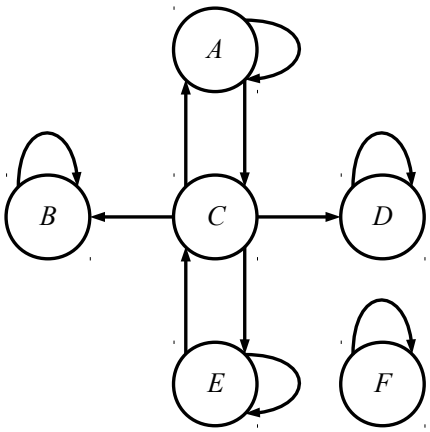
Good luck, and have fun!

Checkpoint Questions Due Monday, January 20th at 11:59PM.
(special later deadline since there is no lecture on Monday)
Remaining Questions Due Friday, January 24th at 2:30PM.

You are required to abide by the Stanford Honor Code. Information about how the Honor Code applies in CS103 can be found in the "CS103 and the Stanford Honor Code" handout available on the course website.

Checkpoint Problem One: Interpersonal Dynamics

The diagram to the right represents a set of people named *A*, *B*, *C*, *D*, *E*, and *F*. If there's an arrow from a person *x* to a person *y*, then person *x* loves person *y*. We'll denote this by writing *Loves*(*x*, *y*). For example, in this picture, we have *Loves*(*C*, *D*) and *Loves*(*E*, *E*), but not *Loves*(*D*, *A*).



There are no "implied" arrows anywhere in this diagram. For example, even though *A* loves *C* and *C* loves *E*, the statement *Loves*(*A*, *E*) is false because there's no direct arrow from *A* to *E*. Similarly, even though *C* loves *D*, the statement *Loves*(*D*, *C*) is false because there's no arrow from *D* to *C*.

Below is a series of ten first-order logic statements. For each of those statements, tell us the minimum number of additional arrows that must be added to the diagram to the right to make that formula true. You're only allowed to add arrows; you can't remove existing ones. If a formula is already true, the answer will be "zero" because no edges need to be added. Then, tell us what those arrows are, and briefly explain why that's the smallest number of arrows that need to be added.

i. $Loves(A, A) \rightarrow Loves(E, E)$

ii. $Loves(A, B) \rightarrow Loves(C, D)$

iii. $Loves(A, C) \rightarrow Loves(C, F)$

iv. $Loves(A, D) \rightarrow Loves(D, E)$

v. $Loves(A, D) \leftrightarrow Loves(B, C)$

vi. $Loves(A, C) \leftrightarrow Loves(E, C)$

vii. $\forall x.\ \exists y.\ Loves(x, y)$

viii. $\forall x.\ \exists y.\ (x \neq y \wedge Loves(x, y))$

ix. $\exists x.\ \forall y.\ Loves(x, y)$

x. $\exists x.\ \forall y.\ (x \neq y \rightarrow Loves(x, y))$

Checkpoint Problem Two: First-Order Fundamentals

Consider the following first-order logic statement:

$$\exists p.\ (Problem(p) \wedge \forall g.\ (Program(g) \rightarrow \neg Solves(g, p)))$$

Here, the predicate *Problem*(*x*) states that *x* is a problem, the predicate *Program*(*y*) states that *y* is a program, and the predicate *Solves*(*a*, *b*) states that *a* solves *b*.

i. Translate this formula into plain English. No justification is required.

ii. Write a first-order logic formula that is the negation of the above formula. Your formula should not include any negations, except possibly for direct negations of predicates.

*We **strongly** recommend reading over the Guide to Negations before attempting the above problem.*

iii. Using only the predicates given above, write a statement in first-order logic that says "there is a program that solves every problem." (Alas, this isn't true, but wouldn't it be nice?)

*We **strongly** recommend reading over the Guide to Logic Translations before attempting the above problem.*

Problem One: Propositional Completeness

In this problem, you'll explore some redundancies within the language of propositional logic.

This problem is autograded and should be submitted as part of the coding component for this problem set. Download the starter files for Problem Set Two from the course website, extract them somewhere convenient, and edit `PropositionalCompleteness.proplogic` with your answers. There's information inside each file with information about how to structure your answer. Briefly, if the online Truth Table Tool can understand your answer, so can our autograder. As usual, feel free to submit as many times as you'd like; we'll only grade your last submission.

In lecture, we covered the seven propositional connectives, which for convenience we've listed below:

$$\wedge \quad \vee \quad \neg \quad \rightarrow \quad \leftrightarrow \quad \top \quad \bot$$

We settled on this set of connectives because they're convenient and expressive. However, it turns out that we could get away with fewer connectives than this.

i. Write expression equivalent to $\bot$ that does not use any connectives besides $\wedge$, $\vee$, $\neg$, and $\top$. (You're welcome to use parentheses, but should not use any variables.)

ii. Write an expression equivalent to $p \rightarrow q$ that does not use any connectives besides $\wedge$, $\vee$, $\neg$, and $\top$. (You're welcome to use the variables p and q, along with parentheses.)

iii. Write an expression equivalent to $p \leftrightarrow q$ that does not use any connectives besides $\wedge$, $\vee$, $\neg$, and $\top$. (You're welcome to use the variables p and q, along with parentheses.)

Your answers to parts (i), (ii), and (iii) of this problem show that the the four propositional connectives $\wedge$, $\vee$, $\neg$, and $\top$ collectively are ***sufficient*** – the other three connectives can be rewritten purely in terms of them. However, there's some redundancy within those four connectives, and we can express all propositional formulas just using three of them.

iv. Write an expression equivalent to $p \vee q$ that does not use any connectives besides $\wedge$, $\neg$, and $\top$. (You're welcome to use the variables p and q, along with parentheses.)

We can push this further. You can rewrite any propositional formula using just the $\rightarrow$ and $\bot$ connectives!

v. Write an expression equivalent to $\top$ that does not use any connectives besides $\rightarrow$ and $\bot$. (You're welcome to use parentheses, but should not use any variables.)

vi. Write an expression equivalent to $\neg p$ that does not use any connectives besides $\rightarrow$ and $\bot$. (You're welcome to use the variable p, along with parentheses.)

vii. Write an expression equivalent to $p \wedge q$ that does not use any connectives besides $\rightarrow$ and $\bot$. (You're welcome to use the variables p and q, along with parentheses.)

As a hint, what happens if you negate an implication?

To recap: given the $\rightarrow$ and $\bot$ connectives, you can express $\wedge$, $\neg$, and $\top$. From $\wedge$, $\neg$, and $\top$ you can get $\wedge$, $\vee$, $\neg$, and $\top$. And from those four connectives, you can get back the original seven. Overall, any propositional formula can be expressed purely in terms of $\rightarrow$ and $\bot$. Nifty!

Problem Two: Set Theory and Propositional Logic

(Parts (i) – (v) of this question are autograded; please edit the file `SetTheory.proplogic` in the Problem Set Two starter files with your answers. Part (vi) should be submitted as part of your written answers.)

Imagine we have two sets A and B and some object x. Let's introduce two propositional variables:

- a, which states that $x \in A$, and
- b, which states that $x \in B$.

By combining a and b together in different ways using propositional logic, we can express different claims about how x relates to A and B.

i. Write a statement in propositional logic using the variables a and b that says $x \in A \cap B$.

ii. Write a statement in propositional logic using the variables a and b that says $x \in A \cup B$.

iii. Write a statement in propositional logic using the variables a and b that says $x \in A - B$.

iv. Write a statement in propositional logic that says $x \in A \Delta B$. To receive credit, your solution should use at most two connectives.

We care about the total number of connectives you use, not how many different types of connectives you use. For example, the formula $p \wedge q \wedge r \wedge s \rightarrow t$ has four connectives.

Now, suppose we have some third set C. Let's introduce a third propositional variable c that means $x \in C$.

v. Write a statement in propositional logic that says that $x \in A \Delta B \Delta C$. To receive credit, your solution should use at most two connectives. (Note that $A \Delta B \Delta C = (A \Delta B) \Delta C = A \Delta (B \Delta C)$.)

If you've solved part (iv), you should be able to get a solution that uses four connectives. By rewriting parts of the expression, you can then drop down to two.

vi. Using your answer to part (v) as a starting point, simplify the expression $(A \Delta B) \Delta B$ as much as possible. Briefly justify your answer. No formal proof is necessary. *(Please write up your solution to this problem in your written assignment submission, since we want to see your justification in addition to your answer.)*

Generally speaking, if you ever find yourself in possession of a strange set-theoretic expression involving the set combination operations $\cap$, $\cup$, $-$, or Δ, you can use the above technique to get a slightly better handle at what you're looking in fact. In fact, many rules about how these set-theoretic operators work follow directly from propositional logic!

That result in part (vi) is an interesting one. You may want to keep it in mind for later in the quarter. 😃

Problem Three: Executable Logic

This problem, and the ones after it, concern worlds populated by cats, robots, and humans. Love is in the air, and for whatever reason it seems appropriate to pin down the group dynamics using first-order logic. Let's assume we have the predicates

- *Person*(*p*), which states that *p* is a person;
- *Cat*(*c*), which states that *c* is a cat;
- *Robot*(*r*), which states that *r* is a robot; and
- *Loves*(*x*, *y*), which states that *x* loves *y*.

As a note, the *Person*, *Cat*, and *Robot* predicates are mutually exclusive. For example, you can't have a robot cat or a cat person. (I mean, you can have a "cat person" in the sense that you can have a person who likes cats, but not someone who is literally both a cat and a person.) Additionally, you can assume that each entity in the world is either a person, a cat, or a robot. Finally, note that love is not necessarily symmetric. It's possible for *A* to love *B* but not the other way around. (Alas!)

Below is a list of six first-order logic statements. Your task is to implement the six C++ functions defined in the file `ExecutableLogic.cpp`, each of which determines whether one of the first-order logic formulas are true about a set of robots, cats, and people. Each robot, cat, or person is represented using a variable of type `Entity`, and we've provided the following C++ functions to you, which mirror the four predicates given above:

```
bool Person(Entity p);
bool Cat   (Entity c);
bool Robot (Entity r);
bool Loves (Entity x, Entity y);
```

Our provided starter files will run the six functions you'll implement on some sample worlds, which you can use to test out your implementations.

i. Consider the following first-order logic formula: $\exists x.\ Cat(x)$

   Write C++ code for a function

   ```
   bool isFormulaTrueFor_partI(std::set<Entity> S)
   ```

   that takes in a set *S* and returns whether the above formula is true for the entities in that set.

ii. Repeat the above exercise with this first-order logic formula: $\forall x.\ Robot(x)$

iii. Repeat the above exercise with this first-order logic formula: $\exists x.\ (Person(x) \wedge Loves(x, x))$

iv. Repeat the above exercise with this first-order logic formula: $\forall x.\ (Cat(x) \rightarrow Loves(x, x))$

v. Repeat the above exercise with this first-order logic formula:

$$\forall x.\ (Cat(x) \rightarrow \exists y.\ (Person(y) \wedge \neg Loves(x, y)))$$

It's a lot easier to write code for this one if you use a helper function.

vi. Repeat the above exercise with this first-order logic formula:

$$\exists x.\ (Robot(x) \leftrightarrow \forall y.\ Loves(x, y))$$

Problem Four: First-Order Negations

For each of the first-order logic formulas below, find a first-order logic formula that is the negation of the original statement. Your final formula must not have any negations in it except for direct negations of predicates. For example, given the formula

$$\forall c.\ (Cat(c) \rightarrow \exists p.\ (Person(p) \wedge Loves(p, c))),$$

you could give the formula

$$\exists c.\ (Cat(c) \wedge \forall p.\ (Person(p) \rightarrow \neg Loves(p, c))).$$

However, you couldn't give as an answer the formula

$$\exists c.\ (Cat(c) \wedge \neg\exists p.\ (Person(p) \wedge Loves(p, c))),$$

since the inner negation could be pushed deeper into the expression.

To submit your answers, edit the file `FirstOrderNegations.fol` with your final formulas. That file contains information about how to format your answers.

We ***strongly*** recommend reading over the Guide to Negations before starting this problem.

i. $\forall p.\ (Person(p) \rightarrow \exists c.\ (Cat(c) \wedge Loves(p, c) \wedge \forall r.\ (Robot(r) \rightarrow \neg Loves(c, r))))$

ii. $\Big(\forall x.\ \big(Person(x) \leftrightarrow \exists r.\ (Robot(r) \wedge Loves(x, r))\big)\Big) \rightarrow \big(\forall r.\ \forall c.\ (Robot(r) \wedge Cat(c) \rightarrow Loves(r, c))\big)$

Be careful – make sure you understand how that formula is parenthesized before you try negating it.

iii. $\forall c.\ (Cat(c) \rightarrow \exists r.\ (Robot(r) \wedge \forall x.\ (Loves(c, r) \leftrightarrow r = x)))$

iv. $\exists x.\ (Cat(x) \wedge \big(\forall r.\ (Loves(r, x) \rightarrow Robot(r))\big) \vee \forall p.\ \big(Loves(p, x) \rightarrow Person(p)\big)\big))$

Problem Five: This, But Not That

Below is a series of pairs of statements about groups of cats, robots, and people. For each pair, find the ***absolute simplest*** world in which the first statement is true and the second statement is false. (By "absolute simplest," we mean using as few entities as possible, and having as few entities love each other as possible.)

To submit your answers, edit the file `ThisButNotThat.worlds` in the `res/` directory. There's information in that file about how to specify your worlds.

	Make this statement true…	***… and this statement false.***
i.	$\forall y.\ \exists x.\ Loves(x, y)$	$\exists x.\ \forall y.\ Loves(x, y)$
ii.	$\forall x.\ (Person(x) \lor Cat(x))$	$(\forall x.\ Person(x)) \lor (\forall x.\ Cat(x))$
iii.	$(\exists x.\ Robot(x)) \land (\exists x.\ Loves(x, x))$	$\exists x.\ (Robot(x) \land Loves(x, x))$
iv.	$(\forall x.\ Cat(x)) \rightarrow (\forall y.\ Loves(y, y))$	$\forall x.\ \forall y.\ (Cat(x) \rightarrow Loves(y, y))$
v.	$\exists x.\ (Robot(x) \rightarrow \forall y.\ Robot(y))$	$(\forall x.\ Robot(x)) \lor (\forall x.\ \neg Robot(x))$

As a hint, if you want to make a statement false, make its negation true.

That last one is subtle. Does the statement in the left column look fishy to you?

Problem Six: Translating into Logic

In each of the following, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you ***must*** only use the predicates *Person*, *Robot*, *Cat*, and *Loves*.

To submit your answers, edit the file `TranslatingIntoLogic.fol` with your formulas. There's information in that file about the expected format for your answers.

i. Write a statement in first-order logic that says "robots do not love." (How sad!)

As a reminder, love is considered directional. Even if robots do not love, it's possible that people or cats might love robots. For example, I could love my Roomba even if it feels nothing toward me. 😃

ii. Write a statement in first-order logic that says "each robot loves every cat, but no cat loves any person."

iii. Write a statement in first-order logic that says "each cat only loves itself." (Okay, I'm not that cynical about cats. But it's still a good exercise to translate this statement!)

iv. Write a statement in first-order logic that says "if you pick a person, you'll find that they love a cat if and only if they also love a robot."

v. Write a statement in first-order logic that says "each person loves exactly two cats and nothing else." To clarify, each person is allowed to love a different pair of cats.

vi. Write a statement in first-order logic that says "no two robots love exactly the same set of cats."

Looking for a good read on the theme of people, robots, pets, and love? Check out Ted Chiang's novella *The Lifecycle of Software Objects*, which explores these ideas in depth. 😃

Problem Seven: Hereditary Sets

Let's begin with a fun little definition:

A set S is called a ***hereditary set*** if every $x \in S$ is also a hereditary set.

This definition might seem strange because it's self-referential – it defines hereditary sets in terms of other hereditary sets! However, it turns out that this is a perfectly reasonable definition to work with, and in this problem you'll explore some properties of hereditary sets.

i. Given the self-referential nature of the definition of hereditary sets, it's not even clear that hereditary sets even exist at all! As a starting point, prove that there is at least one hereditary set.

When confronted with a new definition, it never hurts to try out some examples. So pick a set, any set, and then apply the above definition to see whether it's hereditary. If so, great! You're done. If not, make sure you can explain why not. Then, based on what you found, pick another set and repeat this process. After a few iterations, you will likely converge on an answer.

*This strategy, by the way, is a **great** way to get a handle on any new definition.*

ii. Prove that if H is a hereditary set, then $\wp(H)$ is also a hereditary set.

The hardest part of this problem is determining how to set this proof up in a way that's precise and rigorous. Here are some things to think about:

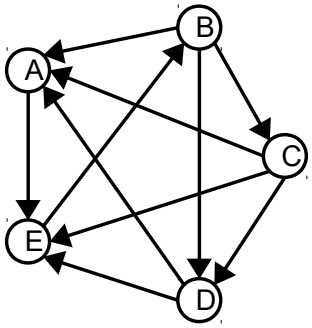
- *What is the general procedure for proving an implication? What's the antecedent here? What's the consequent?*
- *Consider the statement "every $x \in S$ is also a hereditary set." Is that an existentially-quantified statement, a universally-quantified statement, both, or neither? Based on what you know about how to prove existentially-quantified and universally-quantified statements, what should you do to prove this statement?*

After you've written up a draft of your proofs, take a minute to read over them and apply the criteria from the Proofwriting Checklist. Here are a few specific things to watch out for:

- *Although we've just introduced first-order logic as a tool for formalizing definitions and reasoning about mathematical structures, the convention is to **not** use first-order logic notation (connectives, quantifiers, etc.) in written proofs. In a sense, you can think of first-order logic as the stage crew in the theater piece that is a proof – it works behind the scenes to make everything come together, but it's not supposed to be in front of the audience. Make sure that you're still writing in complete sentences, that you're not using symbols like $\forall$ or $\rightarrow$ in place of words like "for any" or "therefore," etc.*
- *A common mistake we see people make when they're just getting started is to restate definitions in the abstract in the middle of a proof. For example, we commonly see people say something like "since $A \subseteq B$, we know that every element of A is an element of B." When you're writing a proof, you can assume that whoever is reading your proof is familiar with the definitions of relevant terms, so statements like the one here that just restate a definition aren't necessary. Instead of restating definitions, try to apply those definitions. A better sentence would be something to the effect of "Since $x \in A$ and $A \subseteq B$, we see that $x \in B$," which uses the definition to conclude something about a specific variable rather than just restating the definition. See the Guide to Proofs on Set Theory for more details.*

Hereditary sets are used in ***foundational mathematics***, the study of how to assemble all mathematical structures from simple structures. If you'd like to learn more about them, take Math 161!

Problem Eight: Tournament Champions, Part Two



A ***tournament*** is a contest among n players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and adding arrows between pairs of players to indicate who won each game. For example, in the tournament to the right, player A beat player E, but lost to players B, C, and D.

A ***tournament champion*** is a player c where, for each other player p in the tournament, either

- c won her game against p, or
- there's a player q where c won her game against q and q won his game against p.

In the tournament shown here, player B is a champion: she won against A, C, and D, and although she lost to E, she won against D, who in turn won against E. Player C is also a champion: she won against A, D, and E, and the one player she lost to (B) is covered because C won against E, who in turn won against B. Player A, however, is not a tournament champion. To see this, A didn't win against C, nor is there any player who A won against who in turn won against C.

i. Is player D a tournament champion? How about player E? No justification is required.

This is all about applying the definition. You might have an intuition for what a tournament champion is, but to determine the answers to these questions, you'll have to look back at the definition.

Here's an amazing fact about tournaments.

ii. Let T be an arbitrary tournament and p be any player in that tournament. Prove the following statement: if p won more games than anyone else in T or is tied for winning the greatest number of games, then p is a tournament champion in T.

Remember that proofwriting heavily involves leveraging definitions. Intuitively it makes sense that someone who won the most games did the best in a tournament, but that's not what's being asked here. Instead, you're asked to show that if someone won the most games (or tied for winning the most games), then they specifically meet the criteria described above that characterize what a tournament champion is.

*A great question to ponder – what has to happen for someone to **not** be a tournament champion? Don't guess based on your intuition; negate the definition of a tournament champion and see what happens. You might want to draw a picture of what happens in this case.*

A corollary of the result from (ii) is that every tournament with at least one player has at least one tournament champion – pick someone who won the most games or is tied for winning the most games.

iii. Prove or disprove: if T is a tournament and c is a champion in T, then either c won the most games or is tied for winning the most games.

*This is the converse of the result from part (ii). Remember – the converse of an implication is not necessarily equivalent to the original implication. The result you proved in part (ii) of this problem is an example of a style of proof called a **proof by extreme case**. In a proof of that sort, you pick some "extreme" object, usually the biggest or smallest object of some type, then prove that it has some property. Here, you proved that the "winningest" player must be a tournament champion. Keep this idea in mind for the future.*

Next we introduce the notion of a ***loop*** in a tournament. A loop in a tournament T is a series of $n \geq 3$ different players p_1, p_2, …, p_n such that player p_1 won against player p_2, player p_2 won against player p_3, …, and, finally, player p_n won against player p_1. The ***length*** of a loop is the number of players in the loop.

iv. Give an example of a loop of length five in the tournament shown above. No justification is necessary.

v. Prove that if a tournament has any loops at all, then it must have a loop of length three.

This would be a great spot to draw pictures and try out examples. Try drawing tournaments with different numbers of players and loops of different sizes and see if you notice anything. As a hint, use a ***proof by extreme case****. Focus on the smallest loop in the tournament – what can you say about it?*

Optional Fun Problem: Insufficient Connectives (Extra Credit)

Every propositional logic formula could be written in terms of just $\rightarrow$ and $\bot$. However, you *cannot* express every possible propositional logic formula using just the $\leftrightarrow$ and $\bot$ connectives. Prove why not.